Midterm 1 Solutions

1. Compute

$$\int \frac{x^2}{\sqrt{9-x^2}}\,\mathrm{d}x.$$

Solution: Substitute $x = 3\sin\theta$, so

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9-x^2}}\,\mathrm{d}x &= \int \frac{(3\sin\theta)^2}{\sqrt{9-(3\sin x)^2}} 3\cos\theta\,\mathrm{d}\theta \\
&= \int \frac{(3\sin\theta)^2}{\sqrt{9-(3\sin x)^2}} 3\cos\theta\,\mathrm{d}\theta \\
&= 9\int \sin^2\theta\,\mathrm{d}\theta \\
&= 9\int \frac{1}{2}-\frac{1}{2}\cos(2\theta)\,\mathrm{d}\theta \\
&= \frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C \\
&= \frac{9}{2}\left(\theta - \sin(\theta)\cos(\theta)\right) + C \\
&= \frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) - \frac{x}{3}\sqrt{1-\left(\frac{x}{3}\right)^2}\right) + C \\
&= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C
\end{aligned}$$

2. Find the area of the surface obtained by rotating the curve given by

$$y = 2\sqrt{x+1}, \quad 0 \le x \le 3$$

around the x-axis.

Solution: The area is given by

$$\begin{aligned}\int_0^3 2\pi 2\sqrt{x+1}, \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x &= \int_0^3 4\pi\sqrt{x+1}, \sqrt{1+\left(\frac{1}{\sqrt{x+1},}\right)^2}\,\mathrm{d}x \\ &= \int_0^3 4\pi\sqrt{x+2}\,\mathrm{d}x \\ &= 4\pi\frac{2}{3}(x+2)^{\frac{3}{2}}\Big|_0^3 \\ &= \frac{8\pi}{3}(\sqrt{5}^3-\sqrt{2}^3).\end{aligned}$$

3. Find $k \in \mathbb{R}$ such that

$$f(x)=\frac{k}{1+x^2}$$

defines a probability density function.

Solution: Need to find k such that $f(x) \geq 0$ for all x and $\displaystyle\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$.
The first condition is satisfied as long as $k \geq 0$. For the second we compute:

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,\mathrm{d}x = \int_{-\infty}^{0}\frac{1}{1+x^2}\,\mathrm{d}x + \int_{0}^{\infty}\frac{1}{1+x^2}\,\mathrm{d}x.$$

We have

$$\int_0^{\infty}\frac{1}{1+x^2}\,\mathrm{d}x = \lim_{b\to\infty}\int_0^b\frac{1}{1+x^2}\,\mathrm{d}x = \lim_{b\to\infty}\arctan x\Big|_0^b = \lim_{b\to\infty}\arctan b = \frac{\pi}{2},$$

and similarly the other integral is also $\dfrac{\pi}{2}$. Therefore,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = k\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,\mathrm{d}x = \pi k.$$

So f is a probability density function precisely when $k=\dfrac{1}{\pi}$.
Remark: This is called the Cauchy distribution.

4. Compute

$$\int\frac{\mathrm{d}x}{1+\sqrt[3]{x}}.$$

Solution: Substitute $u = \sqrt[3]{x}$ so that $u^3 = x$ and $3u^2\mathrm{d}u = \mathrm{d}x$. Then

$$\int \frac{\mathrm{d}x}{1+\sqrt[3]{x}} = \int \frac{3u^2\mathrm{d}u}{1+u}.$$

Long division gives

$$\frac{3u^2}{1+u} = 3u - 3 + \frac{3}{1+u},$$

hence

$$\begin{aligned}\int \frac{3u^2\mathrm{d}u}{1+u} &= \int 3u - 3 + \frac{3}{1+u}\mathrm{d}u \\ &= \frac{3}{2}u^2 - 3u + 3\ln(u+1) + C = \frac{3}{2}x^{\frac{3}{2}} - 3\sqrt[3]{x} + 3\ln(\sqrt[3]{x}+1) + C\end{aligned}$$

(Perhaps a little quicker would be substituting $u = 1 + \sqrt[3]{x}$, then the denominator is u and the division is easier)

5. For each of the following improper integrals decide whether it converges or diverges. No justification is needed.

(Correct answer = +3 points, wrong answer = 0 points, blank = 1.5 points)

Integral	Converges	Diverges
(a) $\displaystyle\int_0^\infty x^{20}e^{-7x}\,\mathrm{d}x$		
(b) $\displaystyle\int_{-1}^1 \frac{1}{x}\,\mathrm{d}x$		
(c) $\displaystyle\int_{-\infty}^\infty \frac{e^{-x}}{1+x^2}\,\mathrm{d}x$		
(d) $\displaystyle\int_0^\infty \frac{e^x}{1+e^{2x}}\,\mathrm{d}x$		
(e) $\displaystyle\int_0^1 \frac{1+x^2}{1-x^2}\,\mathrm{d}x$		
(f) $\displaystyle\int_1^\infty \frac{x}{1+x^3}\,\mathrm{d}x$		
(g) $\displaystyle\int_1^\infty \frac{\arctan\left(\frac{1}{x}\right)}{x}\,\mathrm{d}x$		

Solution:

(a) Converges. Intuition: Exponential goes to 0 much quicker than the polynomial grows. To see it rigorously either integrate by parts 20 times to get rid of the x^{20} term or use $\lim_{x\to\infty}\frac{x^{20}}{e^x} = 0$ to deduce $x^{20} < e^x$ for large enough x, so that we have $x^{20}e^{-7x} \leq e^{-6x}$ for large enough x, and the integral of the latter converges.

(b) Diverges. Split the integral into $\displaystyle\int_{-1}^0 \frac{1}{x}\,\mathrm{d}x + \int_0^1 \frac{1}{x}\,\mathrm{d}x$. The second integral diverges by lecture (the first one also does), so the integral in question diverges.

(c) Diverges. Intuition: As $x \to -\infty$, the integrand goes to ∞.

(d) Converges. Intuition: For large x, $\dfrac{e^x}{1+e^{2x}}$ is approximately $\dfrac{e^x}{e^{2x}} = e^{-x}$, the integral of which converges. Precisely: $\dfrac{e^x}{1+e^{2x}} < \dfrac{e^x}{e^{2x}} = e^{-x}$ and $\displaystyle\int_0^\infty e^{-x}\,\mathrm{d}x$ converges, hence so does the original integral by the comparison test.

(e) Diverges. Intuition: $\dfrac{1+x^2}{1-x^2} = \dfrac{1+x^2}{(1-x)(1+x)} \approx \dfrac{1}{1-x}$ for x close to 0 and the integral of $\dfrac{1}{1-x}$ over $[0,1]$ diverges by lecture (it is basically the same as $\displaystyle\int_0^1 \frac{1}{x}\,\mathrm{d}x$ after a substitution). Precisely: Do comparison test using $\dfrac{1+x^2}{1-x^2} > \dfrac{1}{(1-x)(1+x)} > \dfrac{1}{2(1-x)}$.

(f) Converges. Intuition: $\dfrac{x}{1+x^3} \approx \dfrac{x}{x^3} = \dfrac{1}{x^2}$ for large x, and the integral of the latter converges by lecture since $2 > 1$. Precisely: Do comparison test using $\dfrac{x}{1+x^3} > \dfrac{x}{x^3} = \dfrac{1}{x^2}$.

(g) Converges. Intuition: $\arctan z \approx z$ for small z, so $\dfrac{\arctan\left(\frac{1}{x}\right)}{x} \approx \dfrac{1}{x^2}$ for x large, so the integral should converge. Precisely: Use the inequality $\arctan z \leq z$, so that $\dfrac{\arctan\left(\frac{1}{x}\right)}{x} \leq \dfrac{1}{x^2}$ and apply the comparison test.